On the Existence of an Integral Potential in a Weighted Bidirected Graph

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

ABSTRACT

A. Schrijver proved that if A denotes the incidence matrix of a bidirected graph, and b is an integral "length" function on the edges of A, then the system $Ax \leqslant b$ has an integer solution x if and only if (i) each cycle in A has nonnegative length, and (ii) each doubly odd cycle in A has positive length. Unfortunately these cycles may be very complicated. We show that we may restrict conditions (i) and (ii) to a set of reasonably simple cycles.

1. INTRODUCTION

The purpose of this paper is to sharpen the following theorem of [2]. Let $A=(a_{ij})$ be an integral $m\times n$ matrix satisfying

$$\sum_{j=1}^{n}|a_{ij}|=2 \qquad \text{for} \quad i=1,\dots,m, \tag{1}$$

and let b be an integral vector in $\mathbb{R}^m$. Then

$$\begin{array}{ll} \text{the system } Ax\leqslant b \text{ has an integral solution } x \text{ if and only if:} & \\ \quad \text{(i) each cycle in } A \text{ has nonnegative length;} & \quad (2)\\ \quad \text{(ii) each doubly odd cycle in } A \text{ has positive length.} & \end{array}$$

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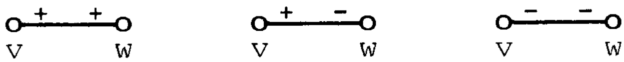
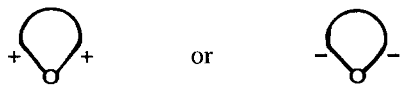
Here the following terminology is used. With each matrix A satisfying (1) we associate a *bidirected graph*, whose *vertices* are the columns of A, and whose *edges* are the rows of A. Edge e is said to *connect* vertices v and w if $|a_{ev}| = |a_{ew}| = 1$. The sign of e at v (respectively w) is positive if a_{ev} (a_{ew}) is positive and negative if a_{ev} (a_{ew}) is negative. We can indicate this as



(3)

An edge e is said to be a *loop* at v if $|a_{ev}| = 2$, indicated as



(4)

A *cycle* in A is a sequence

$$(v_0, e_1, v_1, \ldots, e_d, v_d) \tag{5}$$

such that

$$\begin{aligned}
&\text{(i)} && v_0, v_1, \ldots, v_d \text{ are vertices, with } v_0 = v_d, \text{ and } e_1, e_2, \ldots, e_d \text{ are edges;}\\
&\text{(ii)} && \text{for each } i = 1, \ldots, d, \text{ either } v_{i-1} \neq v_i \text{ and } |a_{e_i v_{i-1}}| = |a_{e_i v_i}| = 1,\\
& && \text{or } v_{i-1} = v_i \text{ and } |a_{e_i v_i}| = 2;\\
&\text{(iii)} && \text{for each } i = 1, \ldots, d,\ a_{e_i v_i} a_{e_{i+1} v_i} < 0
\end{aligned} \tag{6}$$

(taking $e_{d+1} := e_1$). Examples of cycles are given by Figure 1. The *length* of a cycle (5) is, by definition,

$$\sum_{i=1}^{d} b_{e_i} \tag{7}$$

This explains condition (i) in (2).

A cycle (5) is called *doubly odd* if there exists a t with $0 < t < d$ such that

$$\begin{aligned}
&\text{(i)} && v_0 = v_t = v_d;\\
&\text{(ii)} && a_{e_1 v_0} a_{e_t v_t} > 0 \text{ and } a_{e_{t+1} v_t} a_{e_d v_d} > 0;\\
&\text{(iii)} && \textstyle\sum_{i=1}^{t} b_{e_i} \text{ is odd and } \sum_{i=t+1}^{d} b_{e_i} \text{ is odd.}
\end{aligned} \tag{8}$$

Conditions (i) and (ii) are illustrated by Figure 2. This explains condition (ii) of (2).

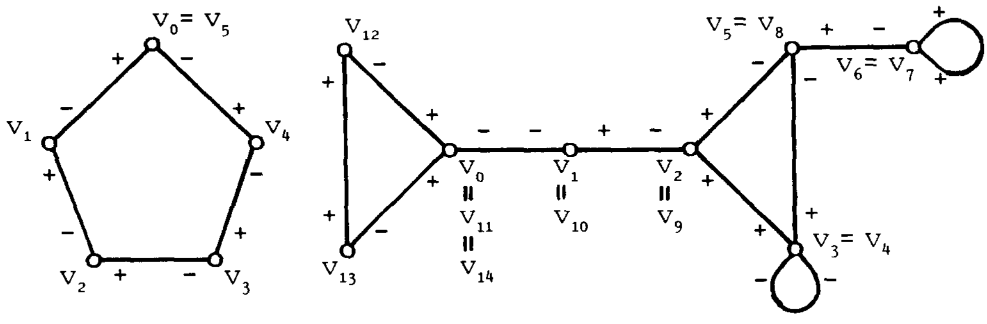


FIG. 1.

It is not sufficient to require condition (2)(i) only for *simple* cycles, i.e. those for which $v_1, \ldots, v_d$ are all distinct. Similarly, condition (2)(ii) cannot be restricted to doubly odd cycles with $v_1, \ldots, v_{d-1}$ all distinct.

Nevertheless, we need not verify (2) for all the cycles in A. In this paper, we describe precisely which cycles of A must be considered. In order to do this, we need the following definition. A cycle (5) is *semisimple* if there exist t and u such that

$$\begin{aligned}
&\text{(i)} && 0 \leqslant u < t - u \leqslant t < d; \\
&\text{(ii)} && v_0 = v_t,\ v_1 = v_{t-1}, \ldots,\ v_u = v_{t-u},\ e_1 = e_t,\ e_2 = e_{t-1}, \ldots, \\
& && e_u = e_{t-u+1}; \\
&\text{(iii)} && v_{u+1},\ v_{u+2}, \ldots, v_{d-1} \text{ are all distinct;} \\
&\text{(iv)} && a_{e_1 v_0} a_{e_t v_t} > 0.
\end{aligned} \tag{9}$$

[Condition (iv) here is superfluous if $u > 0$.]

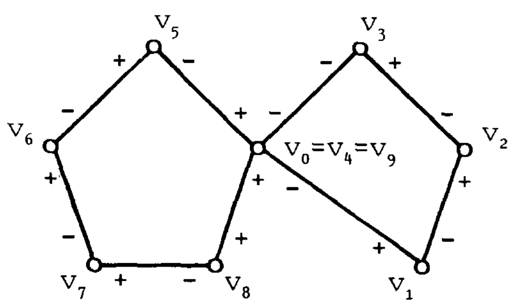


FIG. 2.

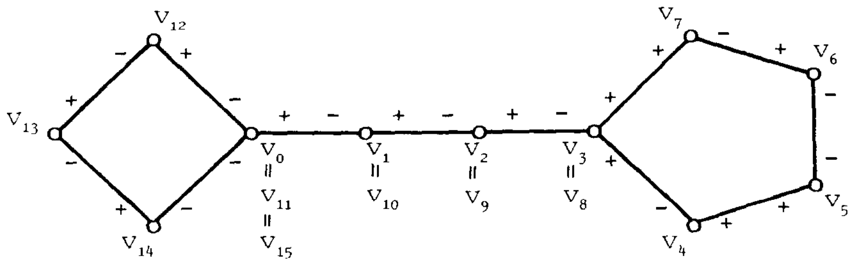


FIG. 3.

As an illustration ($u = 3$, $t = 11$, $d = 15$), consider Figure 3. Here all vertices drawn are distinct. We will show that in (2)(i) we only have to consider simple and semisimple cycles.

Characterizing the cycles necessary in (2)(ii) is more complicated. Call a sequence $P = (v_0, e_1, v_1, \dots, e_d, v_d)$ a *path* if

$$\begin{array}{ll} \text{(i)} & v_0, \dots, v_d \text{ are vertices, and } e_1, \dots, e_d \text{ are edges;} \\ \text{(ii)} & \text{for } i = 1, \dots, d, \text{ either } v_{i-1} \neq v_i \text{ and } |a_{e_i v_{i-1}}| = |a_{e_i v_i}| = 1, \\ & \text{or } v_{i-1} = v_i \text{ and } |a_{e_i v_i}| = 2; \\ \text{(iii)} & \text{for } i = 1, \dots, d-1,\ a_{e_i v_i} a_{e_{i+1} v_i} < 0. \end{array} \tag{10}$$

We allow the case $d = 0$. The *internal vertices* of such a path are $v_1, \dots, v_{d-1}$. The sequence $(v_d, e_d, v_{d-1}, \dots, e_1, v_0)$ is a path again, and is denoted by P^{-1}. If $P' = (v_0', e_1', v_1', \dots, e_{d'}', v_{d'}')$ and $P'' = (v_0'', e_1'', v_1'', \dots, e_{d''}'', v_{d''}'')$ are paths, we say that P' *connects* to P'' if $v_{d'}' = v_0''$ and ($a_{e_{d'}' v_{d'}'} a_{e_1'' v_0''} < 0$ or $d'd'' = 0$). If P' connects to P'' then their *concatenation* $P' \cdot P'' := (v_0', e_1', v_1', \dots, e_{d'}', v_{d'}' = v_0'', e_1'', v_1'', \dots, e_{d''}'', v_{d''}'')$ is again a path.

Call a cycle a *Korach cycle* if there exist paths $P_1, \dots, P_k, Q_1, \dots, Q_{2k}$ (with $k \geqslant 1$) such that the cycle is

$$\begin{aligned} &P_1 \cdot Q_1 \cdot P_2 \cdot Q_2 \cdot P_3 \cdots Q_{k-1} \cdot P_k \cdot Q_k \cdot P_1^{-1} \cdot Q_{k+1} \cdot P_2^{-1} \\ &\quad \cdot Q_{k+2} \cdot P_3^{-1} \cdots Q_{2k-1} \cdot P_k^{-1} \cdot Q_{2k}, \end{aligned} \tag{11}$$

where

(i) all vertices in all $P_1, \dots, P_k$ and all internal vertices in all $Q_1, \dots, Q_{2k}$ are distinct;
(ii) Q_i does not connect to Q_{k+i+1} ($i = 1, \dots, k-1$), and Q_{k+i-1} does not connect to Q_i ($i = 1, \dots, k+1$).

(12)

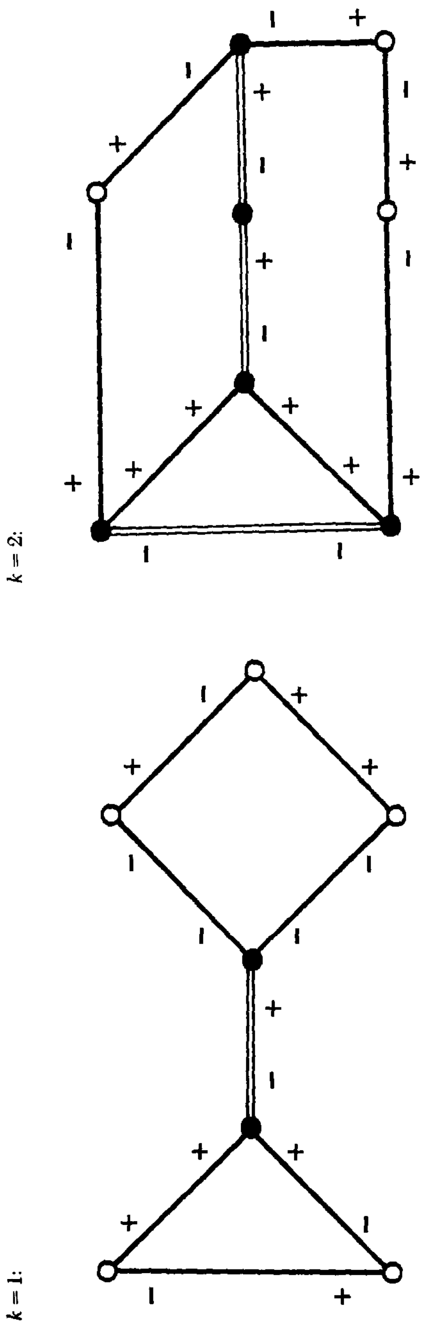
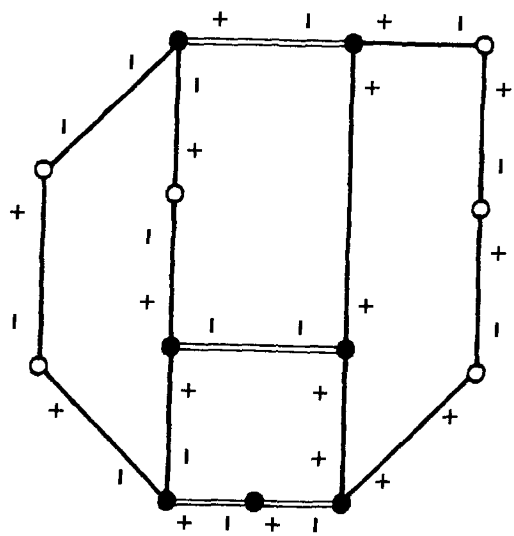


FIG. 4.

$k = 3$:



$k = 4$:

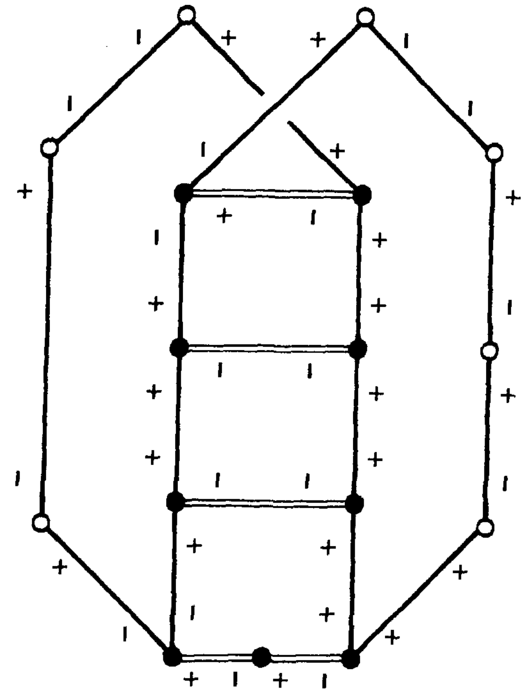


FIG. 4. (Continued)

$k = 5$:

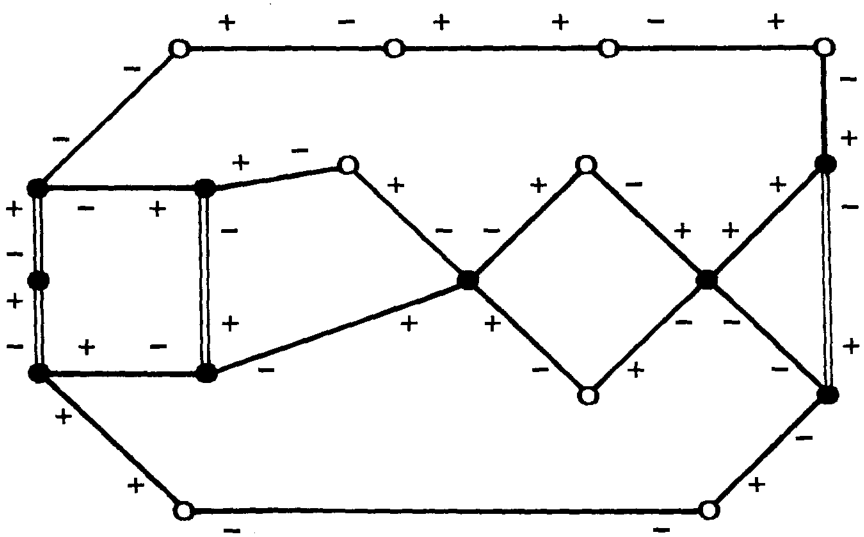


FIG. 4. (*Continued*)

[Note that (12)(i) implies that each Q_i has at least two vertices.] In Figure 4 we give examples for $k = 1, 2, 3, 4, 5$ (all vertices drawn are different, double lines and solid dots denote edges and vertices on the P_i's).
Our theorem is:

THEOREM. *The system $Ax \leqslant b$ has an integer solution x if and only if*

$$\begin{aligned} &\text{(i) each simple or semisimple cycle in } A \text{ has nonnegative length;} \\ &\text{(ii) each doubly odd Korach cycle in } A \text{ has positive length.} \end{aligned} \tag{13}$$

In Section 2 we give a proof of the theorem, based on a theorem of Korach [1]. In Section 3 we show that the conditions (13) cannot be reduced further: each of the cycles described is necessarily included in (13).

2. PROOF OF THE THEOREM

Our proof consists of two parts: first deriving (2)(i) from (13)(i), and second deriving (2)(ii) from (13)(i) and (ii).

I. We derive (2)(i) from (13)(i). Suppose

$$(v_0, e_1, v_1, \ldots, e_d, v_d) \tag{14}$$

is a cycle of negative length. Choose the cycle so that:

$$\begin{array}{ll} \text{(i)} & \text{the number of distinct edges occurring in (14) is as small as possible;} \\ \text{(ii)} & \text{under condition (i), } d \text{ is as small as possible.} \end{array} \tag{15}$$

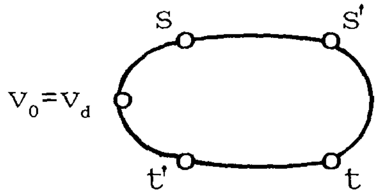
First observe that if $v_s = v_t$ for some $0 < s < t \leqslant d$, then

$$a_{e_s v_s} a_{e_t v_t} < 0. \tag{16}$$

Otherwise $(v_s, e_{s+1}, \ldots, v_t)$ and $(v_t, e_{t+1}, \ldots, v_d = v_0, e_1, \ldots, v_s)$ are cycles, at least one of them with negative length, contradicting (15).

In particular, (16) implies that no three among $v_0, v_1, \ldots, v_{d-1}$ are the same.

We derive that if $v_s = v_t$ and $v_{s'} = v_{t'}$ with $s \neq s'$ and $t \neq t'$ and $0 \leqslant s < t < d$ and $0 \leqslant s' < t' < d$, then we cannot have $s < s' < t < t'$. For suppose that



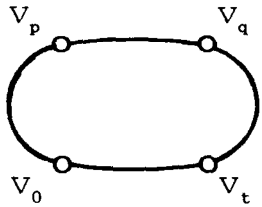
(17)

occurs. By (16), $(v_s, e_{s+1}, \ldots, e_{s'}, v_{s'} = v_{t'}, e_{t'}, \ldots, e_{t+1}, v_t)$ and $(v_{s'}, e_{s'+1}, \ldots, e_t, v_t = v_s, e_s, \ldots, e_1, v_0 = v_d, e_d, \ldots, e_{t'+1}, v_{t'})$ are cycles again, at least one of them with negative length, contradicting (15).

If (14) is not simple, then, by absence of the situation (17), we may assume without loss of generality that v_0 is chosen so that for some t, with $0 < t < d$,

$$v_0 = v_t, \text{ and for } i = t+1, \ldots, d-1, \; v_i \text{ occurs only once in } v_0, v_1, \ldots, v_{d-1}. \tag{18}$$

Then, again by absence of (17), there exist p and q, $0 \leqslant p < q \leqslant t$, such that $v_p = v_q$ and for $i = p+1,\dots,q-1$, v_i occurs only once in $v_0, v_1, \dots, v_{d-1}$:



(19)

(possibly $v_p = v_0$ and $v_q = v_t$). At least one of the cycles

$$(v_0, e_1, v_1, \dots, e_q,\ v_q = v_p, e_p, \dots, e_1,\ v_0 = v_t, e_{t+1}, \dots, v_d)$$

and

$$(v_t, e_t, \dots, e_{p+1},\ v_p = v_q, e_{q+1}, \dots,\ v_t = v_0 = v_d, e_d, \dots, v_t)$$

has negative length and contradicts (15)(i), unless

$$(v_0, e_1, \dots, v_p) \text{ and } (v_t, e_t, \dots, v_q) \text{ are identical paths.} \tag{20}$$

However, if (20) holds, then (14) is semisimple.

Concluding, (14) is simple or semisimple, contradicting (13)(i).

II. We next derive (2)(ii) from (13)(i) and (ii). So by part I of this proof we may assume that (2)(i) holds. Suppose (2)(ii) does not hold. Let

$$C' = (v_0, e_1, v_1, \dots, e_d, v_d) \tag{21}$$

be a doubly odd cycle of length 0. Let t with $0 < t < d$ satisfy (8). We show that there exists a doubly odd Korach cycle of length 0, contradicting (13)(ii). To this end, we may assume that all rows of A occur (as edges) in (21) [we can delete the rows not occurring in (21)].

In order to apply Korach's theorem [1] we construct an auxiliary undirected graph G as follows. For each vertex v of A we have two vertices v^+ and v^-. For each edge (or loop) e of A we make an edge (or loop) e^* in G, where

$$\begin{aligned}
&e^* \text{ connects } v^+ \text{ and } w^+ \text{ if } a_{ev} = a_{ew} = +1,\\
&e^* \text{ connects } v^+ \text{ and } w^- \text{ if } a_{ev} = +1 \text{ and } a_{ew} = -1,\\
&e^* \text{ connects } v^- \text{ and } w^- \text{ if } a_{ev} = a_{ew} = -1, \qquad\qquad (22)\\
&e^* \text{ is a loop at } v^+ \text{ if } a_{ev} = +2,\\
&e^* \text{ is a loop at } v^- \text{ if } a_{ev} = -2.
\end{aligned}$$

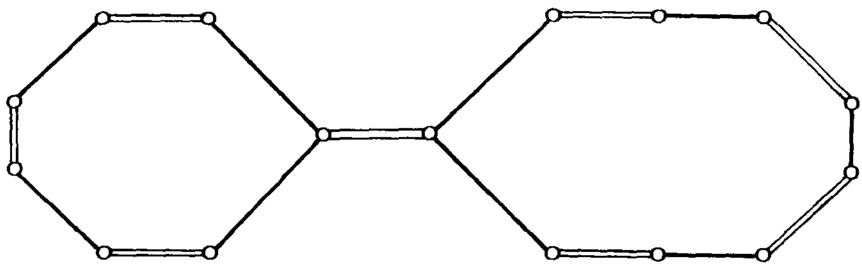


FIG. 5.

Moreover, for each vertex v of A, there is an edge in G connecting v^+ and v^-. These edges form a perfect matching M in G.

Now G does not contain a coclique (= set of pairwise nonadjacent vertices) C of size $|M|$. This follows from the facts that such a coclique C must contain exactly one vertex in every edge in M, and that the doubly odd cycle (21) gives the subgraph of G in Figure 5 (an alternating cycle with respect to M), where double lines stand for edges in M, and single lines for edges not in M. (Vertices and edges drawn different in Figure 5 may coincide.)

Now by Korach's theorem, G contains an alternating cycle of form

$$Q_1 \cdot R_1 \cdot Q_2 \cdot R_2 \dots R_{k-1} \cdot Q_k \cdot R_k \cdot Q_1^{-1} \cdot R_{k+1} \cdot Q_2^{-1}$$
$$\cdot R_{k+2} \cdot Q_3^{-1} \dots R_{2k-1} \cdot Q_k^{-1} \cdot R_{2k}, \tag{23}$$

where $Q_1, \dots, Q_k$, $R_1, \dots, R_{2k}$ are alternating paths with respect to M such that

(24)
- (i) all vertices in all $Q_1, \dots, Q_k$ and all internal vertices in all $R_1, \dots, R_{2k}$ are distinct;
- (ii) the first and last edges of each Q_i belong to M;
- (iii) the first and last edges of each R_i do not belong to M.

[Here, as usual, an *alternating path* with respect to M is a path $(w_0, f_1, w_1, \dots, f_p, w_p)$, where f_i is an edge connecting w_{i-1} and w_i $(i = 1, \dots, p)$, and where exactly one of f_{i-1} and f_i belongs to M $(i = 2, \dots, p)$. The *internal vertices* are $w_1, \dots, w_{p-1}$.]

In a direct way, (23) gives a Korach cycle (11) in A: replace each e^* by e, and replace each $\{v^+, v^-\}$ by v. We call this Korach cycle C''. We show that it is doubly odd and has length 0.

We first show the following:

$$\text{if} \quad y \in \mathbb{R}^m \text{ and } yA = 0 \quad \text{then} \quad yb = 0. \tag{25}$$

To see this, let for any cycle C in A, χ^C denote the incidence vector in $\mathbb{R}^m$ of C, i.e., $\chi^C(e) :=$ the number of times C passes e, for each edge (= row index) of A. By (2), the system $Ax \leqslant 2b$ has a solution x [as condition (2)(ii) is void, since $2b$ is even]. Since $\chi^{C'}b = 0$, $\chi^{C'}A = 0$, and $\chi^{C'} > 0$, we have $Ax = 2b$. This implies (25).

In particular, (25) implies, as $\chi^{C''}A = 0$, that $\chi^{C''}b = 0$, i.e., C'' has length 0.

In order to show that C'', as given by (11), is doubly odd, it suffices to show that

$$\sum_{i=2}^{k} \text{length}(P_i) + \sum_{i=1}^{k} \text{length}(Q_i) \text{ is odd.} \tag{26}$$

This fact follows from

$$\begin{array}{l}\text{if } y \text{ is an integral vector with } yA = (0,\ldots,0,\pm 2,0,\ldots,0),\\ \text{then } yb \text{ is odd.}\end{array} \tag{27}$$

[Applying (27) to the incidence vector of $Q_1 \cdot P_2 \cdot Q_2 \ldots P_k \cdot Q_k$ gives (26).]

To see (27), let yA have its ± 2 at v_s [cf. (21)]. We may assume without loss of generality that $0 < s \leqslant t$. Let z be the incidence vector of the part

$$(v_0, e_1, v_1, \ldots, e_t, v_t) \tag{28}$$

of C'. By (8), $zA = (0,\ldots,0,\pm 2,0,\ldots,0)$, with the ± 2 at v_t, and zb is odd, Let u be the incidence vector of the part $(v_s, e_{s+1}, v_{s+1}, \ldots, e_t, v_t)$ of C'. So $uA = (0,\ldots,0,\pm 1,0,\ldots,0,\pm 1,0,\ldots,0)$, with the ± 1's at v_s and v_t, or $uA = 0$ if $v_s = v_t$. Now

$$(\pm z + 2u \pm y)A = 0, \tag{29}$$

for appropriate (two) choices of $\pm$. Hence by (25), $(\pm z + 2u \pm y)b = 0$. As zb is odd and $2ub$ is even, we know that yb is odd, proving (27).

3. IRREDUNDANCY OF THE CONDITION (13)

We finally show that the condition (13) cannot be reduced any further. Call two cycles *equivalent* if they are the same up to the choice of the starting point, up to the orientation, and, in case they are semisimple, up to replacing $C_1 \cdot C_2$ by $C_1 \cdot C_2^{-1}$. We claim:

> Let A be any bidirected graph. Let C be any simple, semisimple, or Korach cycle of A. Then for some right-hand side b, C is the only cycle of A (up to equivalence) which does not satisfy the condition (13). (30)

To prove our claim, let $C = (v_0, e_1, v_1, \ldots, e_d, v_d)$.

First, let C be a simple or semisimple cycle. Define $b(e) := -2$ for each edge $e \in C$, and $b(e) := 4d + 2$ for each edge $e \notin C$. Then C has negative length. Moreover, each simple or semisimple cycle in A that contains an edge not occurring in C has positive length. We can therefore assume that all edges of A occur in C. Clearly, C is the only simple or semisimple cycle contained in A (up to equivalence), and we should consider C in (13)(i).

Next, let

$$C = P_1 \cdot Q_1 \cdot P_2 \cdot Q_2 \ldots P_k \cdot Q_k \cdot P_1^{-1} \cdot Q_{k+1} \cdot P_2^{-1} \cdot Q_{k+2} \ldots P_k^{-1} \cdot Q_{2k} \tag{31}$$

be a Korach cycle, as in (11). Let $b(e) := (\frac{1}{2}A\mathbf{1})_e$ for $e \in C$, and $b(e) := 2d + 1$ for $e \notin C$. Here $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^n$. One easily checks that now C is doubly odd and has length 0. So $Ax \leqslant b$ has no integer solution. On the other hand, $Ax \leqslant b$ has a rational solution, viz. $x = \frac{1}{2}\mathbf{1}$. So all cycles of A have nonnegative length. Each Korach cycle of A that contains an edge not occurring in C has positive length. Hence we may assume that all edges of A occur in C.

We show that C is the only Korach cycle contained in A (up to equivalence). Define $P_{k+j} := P_j^{-1}$ for $j = 1, \ldots, k$. Let C' be a Korach cycle contained in A. Without loss of generality, we may assume that Q_1 is part of $C' \cdot C'$. Hence also $P_1 \cdot Q_1$ is part of $C' \cdot C'$. Let q be the largest number with $q \leqslant 2k$ such that

$$P_1 \cdot Q_1 \cdot P_2 \cdot Q_2 \ldots P_q \cdot Q_q \tag{32}$$

is part of $C' \cdot C'$. If $q = 2k$, then C' is equivalent to C. This follows from the following fact, which is easy to derive from the definition of a Korach cycle:

$$\text{if } \alpha \text{ is a cycle and is a proper subsequence of } \beta \cdot \beta, \text{ where } \beta \text{ is a Korach cycle, then } \alpha \text{ is equivalent to } \beta. \tag{33}$$

This should be applied to α being cycle (32), and β being C and C'.

So we may assume that $q < 2k$. The part (32) of $C' \cdot C'$ must be followed by P_{q+1}. By the maximality of q, this P_{q+1} cannot be followed by Q_{q+1}, and hence it must be followed by Q_{k+q}^{-1} (taking indices modulo $2k$). That is,

$$P_1 \cdot Q_1 \cdot P_2 \cdot Q_2 \dots P_q \cdot Q_q \cdot P_{q+1} \cdot Q_{k+q}^{-1} \tag{34}$$

is part of $C' \cdot C'$. Since $P_q \cdot Q_q \cdot P_{q+1} \cdot Q_{k+q}^{-1}$ is a cycle, by (33) it is equivalent to C'. Hence $q = 1$ and $k = 1$, and therefore C is equivalent to C'.

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